

GROUP THEORY 2024 - 25, EXERCISE SHEET 9

Exercise 1. (hard) *To always do in every course!*

Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (easy) Let G be a group and fix a prime number p . Show that G has a unique Sylow p -subgroup if and only if all (and hence the only one) of its Sylow p -subgroups are normal.

Exercise 3. (easy) Let $p \neq q$ be two distinct primes and G a finite group of order $p^n q^m$. Suppose that G has a unique Sylow p -subgroup P , as well as a unique Sylow q -subgroup Q . Show that we have an isomorphism of groups $G \cong P \times Q$.

Exercise 4. (easy) Let $\varphi : G \rightarrow H$ be a surjective homomorphism of groups. If $P \subseteq G$ is a Sylow p -subgroup, show that $\varphi(P) \subseteq H$ is a Sylow p -subgroup of H .

Exercise 5. (medium) Find all Sylow 2- subgroups of A_5 .

Exercise 6. (medium) Let G be a finite group and p be a prime number. Let s be a positive integer such that $p^s \mid |G|$. Show that G has a subgroup of order p^s .

Exercise 7. (medium) *Every p -subgroup is contained in a Sylow p -subgroup.*

(1) Let G be a finite group and let $H \subseteq G$ be a subgroup of order p^k for some prime p and $k > 0$. Show that H is contained in a Sylow p -subgroup of G .

Hint: Try to use the idea from the proof of Sylow's second theorem in the lecture notes.

(2) Suppose furthermore that H is a normal subgroup of G with order p^k . Show that H is contained in every Sylow p -subgroup of G .

Exercise 8. (medium) Let K be a normal subgroup of a finite group G and let P be a Sylow p -subgroup of K . Show that G is generated by K and $N_G(P)$.

Exercise 9. (hard) Show that a group G of order 48 is not simple.

Hint: Kernels of group homomorphisms are normal.